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Lagrangian and Hamiltonian formalism on a quantum plane

M Lukin, A Stern and I Yakushin

Department of Physics, University of Alabama, Tuscaloosa, AL 35487, USA

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Abstract. We examine the problem of defining Lagrangian and Hamiltonian mechanics for a particle moving on a quantum plane $Q_{q,p}$. For Lagrangian mechanics, we first define a tangent quantum plane $TQ_{q,p}$ spanned by non-commuting particle coordinates and velocities. Using techniques similar to those of Wess and Zumino, we construct two different differential calculi on $TQ_{q,p}$. These two differential calculi can, in principle, give rise to two different particle dynamics, starting from a single Lagrangian. For Hamiltonian mechanics, we define a phase space $T^*Q_{q,p}$ spanned by non-commuting particle coordinates and momenta. The commutation relations for the momenta can be determined only after knowing their functional dependence on coordinates and velocities. Thus these commutation relations, as well as the differential calculus on $T^*Q_{q,p}$, depend on the initial choice of Lagrangian. We obtain the deformed Hamilton's equations of motion and the deformed Poisson brackets, and their definitions also depend on our initial choice of Lagrangian. We illustrate these ideas for two sample Lagrangians. The first system we examine corresponds to that of a non-relativistic particle in a scalar potential. The other Lagrangian we consider is first order in time derivatives and it is invariant under the action of the quantum group $SL_q(2)$. For that system, $SL_q(2)$ is shown to correspond to a canonical symmetry transformation.

1. Introduction

Recently, quantum groups [1] have attracted much attention in the physics literature partially due to their possible utility in describing particles in two spatial dimensions with generalized spin and statistics [2, 3]. Quantum groups have the property that they act covariantly on generalized spaces, known as quantum planes [4], which are parametrized by non-commuting coordinates. These coordinates have been viewed as the classical limit of 'deformed' creation and annihilation operators [5]. Such creation and annihilation operators have, in turn, been utilized in the construction of quantum group generators [6, 7].

Now if quantum plane coordinates are the classical limit of deformed creation and annihilation operators, how can 'classical' dynamics be introduced on a quantum plane, such that upon canonical quantization, one recovers the deformed creation and annihilation operators? More generally, we can ask how does one define classical Lagrangian and Hamiltonian mechanics on a quantum plane? This is similar to the question of defining classical dynamics for particles moving on spaces parametrized by Grassmann variables; a question which was answered long ago [8].

In this article, we write down the Lagrangian and Hamiltonian formalism for a particle moving on the two-dimensional quantum plane $Q_{q,p}$ [9]. $Q_{q,p}$ is a two-parameter deformation of the 'plane', the parameters being q and p . If x and y denote the non-commuting coordinates of $Q_{q,p}$, then one of the parameters enters in their commutation relation:

$$xy = qyx. \tag{1.1}$$

The other parameter p appears in the differential calculus on $Q_{q,p}$. (The differential calculus for the one-parameter quantum plane was written down by Wess and Zumino [10], and generalized for multiparameter quantum planes in [9].) For this, one introduces one-forms dx and dy , where d denotes an exterior derivative and, as usual, $d^2 = 0$. Unlike the case with commuting coordinates, the exterior product of two one-forms need not be antisymmetric. Instead,

$$dx \wedge dy = -\frac{1}{p} dy \wedge dx. \tag{1.2}$$

The commutation relations (1.1) and (1.2) have the feature that they are unchanged in form under the action of a two-parameter deformation $GL_{q,p}(2)$ of the general linear group in two dimensions. Under the action of $GL_{q,p}(2)$,

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = [T] \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.3}$$

where $[T]$ denotes a 2×2 matrix $[T] = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, whose matrix elements A, B, C and D commute with the coordinates x and y , but do not commute amongst themselves. Rather, they satisfy

$$\begin{aligned} AB &= pBA & AC &= qCA \\ CD &= pDC & BD &= qDB \\ BC &= \frac{q}{p}CB & AD - DA &= \left(q - \frac{1}{p}\right)CB. \end{aligned} \tag{1.4}$$

To completely define the differential calculus on the quantum plane it is necessary to specify the commutation relations between coordinates x and y and their exterior derivatives dx and dy in a consistent manner. According to [9], one can choose

$$\begin{aligned} x dx &= pq dx x \\ x dy &= q dy x + (pq - 1) dx y \\ y dx &= p dx y \\ y dy &= pq dy y. \end{aligned} \tag{1.5}$$

The relations (1.1), (1.2) and (1.5) are consistent in that (i) the exterior derivative of (1.1) agrees with (1.5), that is it follows that from (1.5) that $d(xy - qyx) = 0$ (we assume the usual Leibniz rule for the exterior derivative); (ii) the exterior derivative of the equations (1.5) agrees with equation (1.2); and (iii) no secondary conditions arise from commuting dx or dy through the left-hand side of $xy - qyx = 0$. (Provided $pq \neq -1$, one further finds that $(dx)^2 = (dy)^2 = 0$.) (iv) It can also be checked that like (1.1) and (1.2), the relations (1.5) are preserved under the $GL_{q,p}(2)$ transformations (1.3). (The consistency conditions can be re-expressed in terms of Yang-Baxter equations [10].) (v) Finally we note that the standard differential calculus on a plane is recovered in the limit $q = p = 1$.

(Alternative commutation relations can be found which satisfy (i)-(v). They are obtained by making the replacement $x \rightarrow y, y \rightarrow x, q \rightarrow q^{-1}, p \rightarrow p^{-1}$ in the relations (1.5), while (1.1) and (1.2) remain unchanged. We will not consider the alternative solutions here, but expect that they lead to conclusions which are similar to those we obtain starting from (1.5).)

Now to specify the motion of a particle on $Q_{q,p}$, we need to introduce a velocity vector $v = (\dot{x}, \dot{y})$, and define the commutativity properties of \dot{x} and \dot{y} . We shall do this in section 2. x, y, \dot{x} and \dot{y} coordinatize the ‘tangent quantum plane’ which we shall denote by $TQ_{q,p}$. The Lagrangian will be a function of these variables. In order to write down the Lagrangian formalism, we must be able to perform variations on $TQ_{q,p}$, and consequently define a differential calculus on $TQ_{q,p}$. Thus not only do we need to determine the commutativity properties x, y, \dot{x} and \dot{y} , we also need to determine the commutativity properties of their variations. This will be done in section 3 and in the appendix. We shall require, as usual, that the time derivative commutes with the variational derivative. (For us, time will be a commuting parameter.) In addition, we require that the differential calculus is preserved under the $GL_{q,p}(2)$ transformations (1.3). Two distinct solutions are shown to be consistent with these requirements. In principle, they can lead to two distinct particle dynamics—starting from a single Lagrangian (and differential calculus on $Q_{q,p}$ defined by (1.1), (1.2) and (1.5)).

The Lagrangian and Hamiltonian formalisms are developed in section 4. For the former, we get the usual form for the Euler–Lagrange equations of motion. For the latter, we need to define a phase space $T^*Q_{q,p}$ spanned by the particle coordinates and momenta. Now the commutation relations for the momenta can only be determined after knowing their functional dependence on coordinates and velocities. Therefore these commutation relations depend on the initial choice of Lagrangian. For the Hamiltonian formalism, we must be able to perform variations on $T^*Q_{q,p}$, and thus define a differential calculus on $T^*Q_{q,p}$. The commutation relations associated with this differential calculus will also depend on the initial choice of Lagrangian. We further find that the form of the ‘deformed’ Hamilton’s equations of motion and the ‘deformed Poisson brackets’ will depend on this choice as well. The deformed Poisson brackets coincide with the usual Poisson brackets for commuting variables in the limit $q = p = 1$. Similar deformed Poisson brackets have been postulated by several authors [11], and they are the classical analogue of deformed commutators or ‘quommutators’ appearing in quantum theory. To define the deformed Poisson bracket, we require that the time derivative of any function on $T^*Q_{q,p}$ is the deformed Poisson bracket of that function with the Hamiltonian.

We illustrate the above described formalism by studying two sample Lagrangians. The first Lagrangian we examine is quadratic in time derivatives and, in the limit $q = p = 1$, describes a non-relativistic particle in a scalar potential. The commutativity properties of the coordinates and velocities restrict the possible choices for the potential. The second Lagrangian we examine is first order in time derivatives. It therefore has constraints in the Hamiltonian formalism, and they can be eliminated using the analogue of the Dirac prescription. The Lagrangian is invariant under the action of the quantum group $SL_q(2)$, and these transformations are shown to preserve the Dirac brackets. They therefore correspond to canonical transformations.

We remark in section 5 that by quantizing the second Lagrangian one may indeed obtain deformed creation and annihilation operators. Additional concluding remarks are made in section 5.

2. The tangent quantum plane $TQ_{q,p}$

We wish to describe the dynamics of a particle moving on the quantum plane. For this, let us parametrize the particle trajectory by a (commuting) time coordinate t . Let us also introduce a velocity vector $v = (\dot{x}, \dot{y})$ for the particle, the dot denoting a time derivative.

x, y, \dot{x} and \dot{y} coordinatize the ‘tangent quantum plane’ $TQ_{q,p}$. To fully define $TQ_{q,p}$, we must specify the commutation relations for the velocity components with the coordinates x and y , as well as with themselves. For this, we write the one-forms dx and dy on $TQ_{q,p}$ in terms of the velocity components $dx = \dot{x} dt$ and $dy = \dot{y} dt$. Since we are defining t to be a commuting parameter, the commutation relations of x and y with velocity components \dot{x} and \dot{y} must be the same as the commutation relations of x and y with dx and dy , respectively. That is

$$\begin{aligned} x\dot{x} &= pq\dot{x}x \\ x\dot{y} &= q\dot{y}x + (pq - 1)\dot{x}y \\ y\dot{x} &= p\dot{x}y \\ y\dot{y} &= pq\dot{y}y. \end{aligned} \tag{2.1}$$

These relations are preserved under $GL_{q,p}(2)$ transformations. (The commutation relation between x and \dot{x} presented here differs from that appearing in [12]. The latter commutation relation involves a *non-commuting* constant m associated with the particle mass, as well as the c-number q . It is: $m\dot{x}x = qxm\dot{x}$.)

It remains to specify the commutation relation between \dot{x} and \dot{y} . We can start with the general ansatz: $\dot{x}\dot{y} = c_1yx + c_2\dot{x}x + c_3\dot{y}y + c_4\dot{x}y + c_5\dot{y}x + c_6\dot{y}\dot{x}$. From the requirement that no secondary conditions result from commuting \dot{x} and \dot{y} through equations (2.1), we must have $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ and $c_6 = q$. We are thus left with

$$\dot{x}\dot{y} = q\dot{y}\dot{x}. \tag{2.2}$$

This relation too is preserved under transformations (1.3).

The commutation relations for the coordinates $\{z^1 = x, z^2 = y, z^3 = \dot{x}, z^4 = \dot{y}\}$ of the four-dimensional quantum manifold $TQ_{q,p}$ are specified by equations (1.1), (2.1) and (2.2). We note that these relations *cannot* be written in the simple form: $z^i z^j = q^{ij} z^j z^i, i < j$.

By taking the time derivatives of the relations (2.1) and (2.2), one obtains the commutativity properties for higher-order derivatives of x and y . For example, the derivative of relations (2.1) leads to the commutation relations for components of acceleration with the coordinates

$$\begin{aligned} x\ddot{x} &= pq\ddot{x}x + (pq - 1)\dot{x}^2 \\ x\ddot{y} &= q\ddot{y}x + (pq - 1)(q\dot{y}\dot{x} + \ddot{x}y) \\ y\ddot{x} &= p\ddot{x}y + (pq - 1)\dot{y}\dot{x} \\ y\ddot{y} &= pq\ddot{y}y + (pq - 1)\dot{y}^2 \end{aligned} \tag{2.3}$$

where we have assumed the Leibniz rule holds for time differentiation.

Until now we have examined the commutativity properties of coordinates, velocities, etc, evaluated at a single time t . The same commutativity properties of coordinates, velocities, etc, are assumed to be valid at any other time t' .

The coordinates evaluated at different times do not, in general, commute. To obtain the commutativity properties of two neighbouring points on a particle trajectory associated with two different times $t = t_1$ and $t = t_2$, one may perform a Taylor expansion:

$$\begin{aligned} x(t_2) &= x(t_1) + \dot{x}(t_1)(t_2 - t_1) + \frac{1}{2}\ddot{x}(t_1)(t_2 - t_1)^2 + \dots \\ y(t_2) &= y(t_1) + \dot{y}(t_1)(t_2 - t_1) + \frac{1}{2}\ddot{y}(t_1)(t_2 - t_1)^2 + \dots \end{aligned} \tag{2.4}$$

The commutativity properties of $x(t_2), x(t_1), y(t_2), y(t_1)$ can then, in principle, be determined from conditions (2.1)–(2.3) and their time derivatives. For $pq \neq 1$, these conditions put severe restrictions on the allowable trajectories on $Q_{q,p}$. For the example of a ‘free particle’ $\ddot{x} = \ddot{y} = 0$, equations (2.3) imply that the velocity components are nilpotent

$$\dot{x}^2 = \dot{y}^2 = \dot{x}\dot{y} = 0. \quad (2.5)$$

(Analogous conditions follow for a particle in a potential as we illustrate in section 4.) Further, for the three particle the series (2.4) cuts off and here it is easy to check that the coordinates evaluated at different times do not commute. For instance

$$\begin{aligned} x(t_2)x(t_1) &= x(t_1)^2 + \dot{x}(t_1)x(t_1)(t_2 - t_1) \\ &\neq x(t_1)x(t_2) = x(t_1)^2 + pq\dot{x}(t_1)x(t_1)(t_2 - t_1). \end{aligned} \quad (2.6)$$

To obtain non-trivial particle dynamics it may be desirable to impose that $pq = 1$. In that case, we get no such restrictions (2.5) on a free particle. For the sake of generality however, we shall not set $pq = 1$ in the discussions which follow, except where otherwise stated.

3. Differential calculus on $TQ_{q,p}$

For Lagrangian mechanics we must be able to perform variations on $TQ_{q,p}$, and thus we must define a differential calculus on $TQ_{q,p}$. That is, we need to know the commutation relations between coordinates z^i of $TQ_{q,p}$ and their exterior (or variational) derivatives, which we now denote by δz^i . (We define the exterior derivative δ to be equivalent to d , when it acts on $Q_{q,p} \subset TQ_{q,p}$.) In what follows we find that it is possible to define two different kinds of differential calculus on $TQ_{q,p}$.

As is usual in Lagrangian mechanics, we shall assume that the time derivative commutes with the variational derivative. A few commutation relations necessarily follow from this. They are obtained by comparing the time derivative of (1.5) with the exterior derivative of (2.1). This leads to

$$\begin{aligned} \dot{x}\delta x &= \delta x\dot{x} \\ \dot{y}\delta y &= \delta y\dot{y} \\ x\delta\dot{x} &= pq\delta\dot{x}x + (pq - 1)\delta x\dot{x} \\ y\delta\dot{y} &= pq\delta\dot{y}y + (pq - 1)\delta y\dot{y} \end{aligned} \quad (3.1)$$

where we assume the Leibniz rule for δ and $pq \neq -1$. In addition to (3.1), one obtains the identities

$$\begin{aligned} \dot{y}\delta x + y\delta\dot{x} &= p(\delta\dot{x}y + \delta x\dot{y}) \\ \dot{y}\delta x + p\dot{x}\delta y &= p\delta x\dot{y} + \delta y\dot{x} \\ \dot{x}\delta y + x\delta\dot{y} &= q(\delta\dot{y}x + \delta y\dot{x}) + (qp - 1)(\delta\dot{x}y + \delta x\dot{y}). \end{aligned} \quad (3.2)$$

By further taking the exterior derivative of (3.1) and (3.2), we get commutation relations between one-forms δz^i on $TQ_{q,p}$:

$$\begin{aligned} \delta x \wedge \delta\dot{x} &= -\delta\dot{x} \wedge \delta x \\ \delta y \wedge \delta\dot{y} &= -\delta\dot{y} \wedge \delta y \end{aligned} \quad (3.3)$$

along with the identity

$$\delta x \wedge \delta \dot{y} + \frac{1}{p} \delta y \wedge \delta \dot{x} = -\delta \dot{x} \wedge \delta y - \frac{1}{p} \delta \dot{y} \wedge \delta x. \quad (3.4)$$

As in the previous section, we have assumed that $\delta^2 = 0$ and $(\delta z^i)^2 = 0$.

To completely define the differential calculus on $TQ_{q,p}$, we shall make ansätze for the remaining commutation relations. Our ansätze are such that all terms involve the same number of velocities and coordinates of $Q_{q,p}$.

We start with $\delta \dot{x} \wedge \delta \dot{y} = r \delta \dot{y} \wedge \delta \dot{x}$, r being a c-number. Invariance under $GL_{q,p}(2)$ transformations (1.3) immediately fixes r to be either $r = q$ or $r = -1/p$. With the former solution, however, we do not get the usual antisymmetric exterior product in the limit $q = p = 1$. We shall therefore not consider this case. We are thus left with

$$\delta \dot{x} \wedge \delta \dot{y} = -\frac{1}{p} \delta \dot{y} \wedge \delta \dot{x}. \quad (3.5)$$

Equations (1.2), (3.3) and (3.5) give some of the commutation relations between two one-forms δz^i on $TQ_{q,p}$. For the remaining such relations, we define a 2×2 matrix

$[f] = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$, with the elements f_{ij} being c-numbers, and

$$\begin{pmatrix} \delta x \wedge \delta \dot{y} \\ \delta y \wedge \delta \dot{x} \end{pmatrix} = [f] \begin{pmatrix} \delta \dot{y} \wedge \delta x \\ \delta \dot{x} \wedge \delta y \end{pmatrix}. \quad (3.6)$$

The four matrix elements of $[f]$ can be determined (i) by demanding that no secondary conditions on combinations of δz^i result from commuting δz^i through the relation (1.2); (ii) by imposing the identity (3.4); and (iii) by demanding that the relations (3.6) are preserved under the $GL_{q,p}(2)$ transformations (1.3). We now proceed with (i)–(iii):

(i) By multiplying $\delta \dot{x}$ on the right of $\delta x \wedge \delta y + (1/p)\delta y \wedge \delta x = 0$, we find

$$f_{12}f_{21} = 0. \quad (3.7)$$

(ii) Upon substituting the ansatz (3.6) into the identity (3.4), we get the conditions

$$pf_{11} + f_{21} = -1 \quad \text{and} \quad pf_{12} + f_{22} = -p. \quad (3.8)$$

(iii) Invariance under $GL_{q,p}(2)$ transformations is insured provided

$$f_{11} + qf_{12} = -q \quad pf_{11} - qf_{22} = 0 \quad \text{and} \quad (qp - 1)f_{11} - qf_{12} + qf_{21} = 0. \quad (3.9)$$

For general q and p , there are two possible solutions to equations (3.7)–(3.9), which we shall call cases (a) and (b). The case (a) commutation relations are

$$\begin{aligned} \delta x \wedge \delta \dot{y} &= -\frac{1}{p} \delta \dot{y} \wedge \delta x + \frac{1}{pq} (1 - pq) \delta \dot{x} \wedge \delta y \\ \delta y \wedge \delta \dot{x} &= -\frac{1}{q} \delta \dot{x} \wedge \delta y \end{aligned} \quad (3.10)$$

while for case (b) we have

$$\begin{aligned} \delta x \wedge \delta \dot{y} &= -q \delta \dot{y} \wedge \delta x \\ \delta y \wedge \delta \dot{x} &= (pq - 1) \delta \dot{y} \wedge \delta x - p \delta \dot{x} \wedge \delta y. \end{aligned} \quad (3.11)$$

Finally, eight commutation relations remain to be specified in order to completely fix the calculus. They are the commutation relations between coordinates z^i of $TQ_{q,p}$ and one-forms δz^i on $TQ_{q,p}$ not already contained in equations (1.5) and (3.1). A similar procedure to that used to obtain relations (3.10) and (3.11) can be employed to write down four of these remaining relations. We carry this out in the appendix. Like with (3.10) and (3.11), we find two distinct solutions, and they correspond to cases (a) and (b). For case (a) we find

$$\begin{aligned}x\delta y &= q\delta yx + (pq - 1)(\delta xy + \delta x\dot{y}) \\y\delta x &= p\delta xy + (pq - 1)\delta y\dot{x} \\ \dot{x}\delta y &= q\delta y\dot{x} \\ \dot{y}\delta x &= (1 - pq)\delta y\dot{x} + p\delta x\dot{y}\end{aligned}\tag{3.12}$$

and for case (b) we find

$$\begin{aligned}x\delta y &= q\delta yx + (pq - 1)\left(\delta xy + \frac{1}{p}\delta y\dot{x}\right) + \left(pq + \frac{1}{pq} - 2\right)\delta x\dot{y} \\y\delta x &= p\delta xy + \frac{1}{q}(pq - 1)\delta x\dot{y} \\ \dot{x}\delta y &= \frac{1}{p}\delta y\dot{x} + \frac{1}{pq}(pq - 1)\delta x\dot{y} \\ \dot{y}\delta x &= \frac{1}{q}\delta x\dot{y}.\end{aligned}\tag{3.13}$$

Both sets of relations (3.12) and (3.13) satisfy the identities (3.2), and are preserved under the $GL_{q,p}(2)$ transformations (1.3). By taking the exterior derivative of equations (3.12) and (3.13), we recover the relations (3.10) and (3.11), respectively.

The final four commutation relations are between velocities \dot{x} and \dot{y} and their exterior derivatives $\delta\dot{x}$ and $\delta\dot{y}$. To obtain them we first note that the commutation relation between the two coordinates x and y of $Q_{q,p}$ is the same as that between the two velocities \dot{x} and \dot{y} , and the commutation relation between the one-forms δx and δy is the same as that between $\delta\dot{x}$ and $\delta\dot{y}$. Then from (1.5), a self-consistent set of commutation relations between velocities and their exterior derivatives is

$$\begin{aligned}\dot{x}\delta\dot{x} &= pq\delta\dot{x}\dot{x} \\ \dot{x}\delta\dot{y} &= q\delta\dot{y}\dot{x} + (pq - 1)\delta\dot{x}\dot{y} \\ \dot{y}\delta\dot{x} &= p\delta\dot{x}\dot{y} \\ \dot{y}\delta\dot{y} &= pq\delta\dot{y}\dot{y}.\end{aligned}\tag{3.14}$$

It can also be checked that these relations are consistent with (3.12) and (3.13), in that no secondary conditions arise from commuting z^i through (3.14).

We have thus found two consistent differential calculi on $TQ_{q,p}$: cases (a) and (b). The results can be summarized in terms of the 16×16 R -matrix appearing in the relation

$$z^i\delta z^j = R^{ij}_{kl}\delta z^k z^l\tag{3.15}$$

where we again denote $\{z^1 = x, z^2 = y, z^3 = \dot{x}, z^4 = \dot{y}\}$. By taking the exterior derivative of (3.15), we get

$$\delta z^i \wedge \delta z^j = -R^{ij}_{kl}\delta z^k \wedge \delta z^l.\tag{3.16}$$

R^{ij}_{kl} satisfies the Yang–Baxter equations [10]

$$R^{ab}_{ij}R^{jc}_{kf}R^{ik}_{de} = R^{bc}_{ij}R^{ai}_{dk}R^{kj}_{ef}. \quad (3.17)$$

For the case (a) solution the non-vanishing components of R are

$$\begin{aligned} R^{12}_{12} &= R^{13}_{13} = R^{14}_{14} = R^{23}_{23} = R^{24}_{24} = R^{34}_{34} = R^{14}_{32} = -R^{41}_{23} = pq - 1 \\ R^{11}_{11} &= R^{22}_{22} = R^{33}_{33} = R^{44}_{44} = R^{13}_{31} = R^{24}_{42} = pq \\ R^{12}_{21} &= R^{34}_{43} = R^{14}_{41} = R^{32}_{23} = q \\ R^{21}_{12} &= R^{43}_{34} = R^{41}_{14} = R^{23}_{32} = p \\ R^{31}_{13} &= R^{42}_{24} = 1. \end{aligned} \quad (3.18)$$

For the case (b) solution the non-vanishing components of R are

$$\begin{aligned} R^{12}_{12} &= R^{13}_{13} = R^{24}_{24} = R^{34}_{34} = R^{14}_{32} = pR^{14}_{23} = qR^{23}_{14} = pqR^{32}_{14} = pq - 1 \\ R^{11}_{11} &= R^{22}_{22} = R^{33}_{33} = R^{44}_{44} = R^{13}_{31} = R^{24}_{42} = pq \\ R^{12}_{21} &= R^{34}_{43} = R^{14}_{41} = (R^{41}_{14})^{-1} = q \\ R^{21}_{12} &= R^{43}_{34} = R^{23}_{32} = (R^{32}_{23})^{-1} = p \\ R^{14}_{14} &= pq + \frac{1}{pq} - 2 \\ R^{31}_{13} &= R^{42}_{24} = 1. \end{aligned} \quad (3.19)$$

The relations (3.15) and (3.16) are unchanged under the following action of $GL_{q,p}(2)$: $z^i \rightarrow z'^i = t^i_j z^j$ and $\delta z^i \rightarrow \delta z'^i = t^i_j \delta z^j$, where

$$[t] = \begin{pmatrix} [T] & 0 \\ 0 & [T] \end{pmatrix}. \quad (3.20)$$

$[T]$ once again is the 2×2 matrix $[T] = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with non-commuting matrix elements. A , B , C and D have the commutativity properties given in equation (1.4). The condition of invariance can be stated as follows

$$R^{ij}_{cd} t^a_i t^b_j = R^{ab}_{ij} t^c_a t^d_b. \quad (3.21)$$

4. Lagrangian and Hamiltonian mechanics on $Q_{q,p}$

In order to define a Lagrangian formalism for particles moving on $Q_{q,p}$ we need to take partial derivatives with respect to the coordinates of the tangent quantum plane $TQ_{q,p}$. Here we shall work exclusively in terms of *right* derivatives. If K is a function on the quantum plane $Q_{q,p}$, we define the right derivatives of K by writing variations δK of K according to $\delta K(x, y) = \delta x \partial K / \partial x + \delta y \partial K / \partial y$. The Lagrangian L should be a function on the tangent plane $TQ_{q,p}$. Right derivatives of L are then defined by

$$\delta L(x, y, \dot{x}, \dot{y}) = \delta x \frac{\partial L}{\partial x} + \delta y \frac{\partial L}{\partial y} + \delta \dot{x} \frac{\partial L}{\partial \dot{x}} + \delta \dot{y} \frac{\partial L}{\partial \dot{y}}. \quad (4.1)$$

An action principle can now be formulated on $TQ_{q,p}$, and it leads to the standard form for the Euler–Lagrange equations (where we interpret all derivatives with respect to the coordinates of $TQ_{q,p}$ as right derivatives). If we define an action $S = \int dt L(x, y, \dot{x}, \dot{y})$, then we obtain

$$\dot{\pi}_x - \partial L / \partial x = \dot{\pi}_y - \partial L / \partial y = 0 \quad (4.2)$$

when we ‘extremize’ the action, that is set $\delta S = 0$ with respect to x and y . π_x and π_y are the canonical momenta associated with x and y respectively, where we assume the usual definition

$$\pi_x = \partial L / \partial \dot{x} \quad \text{and} \quad \pi_y = \partial L / \partial \dot{y}. \quad (4.3)$$

The solutions of the equations of motion (4.2) must be consistent with the commutation relations (1.1) and (2.1)–(2.3), and as we will see in example 1, this is not always possible.

In order to pass to the Hamiltonian formalism we need to define a phase space $T^*Q_{q,p}$. It should be spanned by the variables x , y , π_x and π_y . We thus need to know the commutativity properties of the coordinates and momenta. But the commutativity properties of the momenta can only be determined once we know how to write π_x and π_y in terms of x , y , \dot{x} and \dot{y} from equations (4.3). For this we must know the functional form of L . So the commutativity properties of the phase space variables are dynamically determined from the initial choice of Lagrangian.

To write down Hamilton’s equations of motion for the system, we further need the differential calculus on $T^*Q_{q,p}$, and this too can only be determined after knowing π_x and π_y in terms of x , y , \dot{x} and \dot{y} . To be more explicit, let us define the Hamiltonian according to

$$H = \dot{x}\pi_x + \dot{y}\pi_y - L \quad (4.4)$$

and note from (4.1) and (4.2) that variations δH of H can be written

$$\delta H = -\delta x \frac{\partial L}{\partial x} - \delta y \frac{\partial L}{\partial y} + \dot{x}\delta\pi_x + \dot{y}\delta\pi_y. \quad (4.5)$$

In terms of right derivatives of H , we can also express the variations δH according to

$$\delta H = \delta x \frac{\partial H}{\partial x} + \delta y \frac{\partial H}{\partial y} + \delta\pi_x \frac{\partial H}{\partial \pi_x} + \delta\pi_y \frac{\partial H}{\partial \pi_y}. \quad (4.6)$$

Hamilton’s equations of motion are standardly obtained by equating the right-hand sides of equations (4.5) and (4.6), and by assuming independent variations δx , δy , $\delta\pi_x$ and $\delta\pi_y$. But for this we need to know the commutativity properties of \dot{x} with $\delta\pi_x$ and \dot{y} with $\delta\pi_y$. Thus the form of Hamilton’s equations of motion is dependent on the initial choice of Lagrangian. Moreover, the results may be different for the case (a) and the case (b) commutation relations. In general, we cannot even conclude from (4.2), (4.5) and (4.6) that $\dot{\pi}_x = -\partial H / \partial x$ or $\dot{\pi}_y = -\partial H / \partial y$.

Also dependent on the choice of L is the form of the ‘deformed Poisson bracket’ $\{, \}_{q,p}$. This must be true if one requires that the time evolution of any function \mathcal{F} of x , y , π_x and π_y (and possibly t) should be determined from the equation

$$\dot{\mathcal{F}} = \{\mathcal{F}, H\}_{q,p} + \frac{\partial \mathcal{F}}{\partial t}. \quad (4.7)$$

For us, equation (4.7) defines the deformed Poisson bracket. For arbitrary values of q and p , this definition will, in general, differ from the usual definition of the Poisson bracket as given in terms of partial derivatives with respect to the phase space variables.

We next illustrate these ideas with two examples of Lagrangian systems on $TQ_{q,p}$.

4.1. Example 1

The first Lagrangian we consider is second order in time derivatives and corresponds to a deformation of the non-relativistic particle (with mass equal to one) in a scalar potential $V = V(x, y)$. It is

$$L = \frac{1}{1 + pq}(\dot{x}^2 + \dot{y}^2) - V(x, y). \quad (4.8)$$

(We remark here that we are allowed to introduce arbitrary constant coefficients in the Lagrangian, such as the particle mass. The coefficients may be either c-numbers or have non-trivial commutativity properties. This is unlike the system examined in [12] where the commutativity properties of coordinates and momenta along with the equations of motion required the mass to be non-commuting.) Using (3.14), variations δL of L are

$$\begin{aligned} \delta L &= \frac{1}{1 + pq}(\delta\dot{x}\dot{x} + \dot{x}\delta\dot{x} + \delta\dot{y}\dot{y} + \dot{y}\delta\dot{y}) - \delta x \frac{\partial V}{\partial x} - \delta y \frac{\partial V}{\partial y} \\ &= \delta\dot{x}\dot{x} + \delta\dot{y}\dot{y} - \delta x \frac{\partial V}{\partial x} - \delta y \frac{\partial V}{\partial y} \end{aligned} \quad (4.9)$$

which leads to the usual form for the Euler-Lagrange equations for a non-relativistic particle in a scalar potential

$$\ddot{x} = -\partial V/\partial x \quad \text{and} \quad \ddot{y} = -\partial V/\partial y. \quad (4.10)$$

For arbitrary values of q and p , the commutation relations (2.3) put strong constraints on the allowable solutions to the equations of motion. In section 2, we saw that they led to only trivial solutions to the equations for a free particle. Similar conclusions can be drawn for non-zero potentials V . Equations (2.3) lead to a condition on V itself:

$$q \frac{\partial V}{\partial y} x - x \frac{\partial V}{\partial y} = \frac{\partial V}{\partial x} y - qy \frac{\partial V}{\partial x}. \quad (4.11)$$

When $q = p = 1$, this is satisfied for any function V which commutes with both x and y .

A V which satisfies (4.11) for arbitrary values of q and p , is the harmonic oscillator potential

$$V(x, y) = \frac{x^2 + y^2}{1 + pq}. \quad (4.12)$$

The only other potentials satisfying (4.11) for $pq \neq 1$ are those obtained by multiplying x^2 and y^2 in (4.12) by c-number coefficients. (More possibilities arise when one introduces non-commuting constants in the theory, as in [12].) These systems have no non-trivial solutions because the remaining conditions in (2.3) impose additional constraints between the velocities and coordinates. For the potential (4.12) we get $\dot{x}^2 = x^2$, $\dot{y}^2 = y^2$ and $\dot{x}\dot{y} = xy$.

For the case of $qp = 1$, non-trivial solutions to the equations of motion (4.10) are possible. In that case, all of equations (2.3) can be written as

$$\begin{aligned} x \frac{\partial V}{\partial x} &= \frac{\partial V}{\partial x} x & x \frac{\partial V}{\partial y} &= q \frac{\partial V}{\partial y} x \\ y \frac{\partial V}{\partial y} &= \frac{\partial V}{\partial y} y & qy \frac{\partial V}{\partial x} &= \frac{\partial V}{\partial x} y \end{aligned} \quad (4.13)$$

and there are no further constraints on the velocities and coordinates. Conditions (4.13) are, for instance, satisfied for the potential (4.12). When $pq = 1$, the commutation relations of the previous sections simplify significantly and cases (a) and (b) coincide. For example 1 we will see that Hamilton's equations of motion and the Poisson brackets have the usual form when $qp = 1$.

From equation (4.3), the canonical momenta π_x and π_y are identified with the velocity components

$$\pi_x = \dot{x} \quad \text{and} \quad \pi_y = \dot{y}. \tag{4.14}$$

This then defines the commutativity properties of the phase-space variables, and also defines the differential calculus on $T^*Q_{q,p}$, since the commutation relations for π_x and π_y (and their variations) must be identical to the commutation relations for \dot{x} and \dot{y} (and their variations), respectively.

From (4.4), the Hamiltonian for the system is

$$H = \frac{pq}{1 + pq}(\pi_x^2 + \pi_y^2) + V(x, y). \tag{4.15}$$

Variations δH of H can now be written

$$\delta H = -\delta x \frac{\partial L}{\partial x} - \delta y \frac{\partial L}{\partial y} + pq\delta\pi_x\dot{x} + pq\delta\pi_y\dot{y} \tag{4.16}$$

where we have used (3.14) and (4.5). By comparing equation (4.16) with (4.6), we then get the following Hamilton's equations of motion

$$\begin{aligned} \dot{\pi}_x &= -\frac{\partial H}{\partial x} & \dot{x} &= \frac{1}{pq} \frac{\partial H}{\partial \pi_x} \\ \dot{\pi}_y &= -\frac{\partial H}{\partial y} & \dot{y} &= \frac{1}{pq} \frac{\partial H}{\partial \pi_y} \end{aligned} \tag{4.17}$$

Using H given in equation (4.15) we can verify that Hamilton's equations of motion (4.17) for the system are identical to the Euler-Lagrange equations (4.10). Equations (4.17) reduce to the usual form for Hamilton's equations of motion when $qp = 1$.

If \mathcal{F} is an arbitrary function of x, y, π_x, π_y and t , its time derivative can then be written according to

$$\begin{aligned} \dot{\mathcal{F}} &= \dot{x} \frac{\partial \mathcal{F}}{\partial x} + \dot{y} \frac{\partial \mathcal{F}}{\partial y} + \dot{\pi}_x \frac{\partial \mathcal{F}}{\partial \pi_x} + \dot{\pi}_y \frac{\partial \mathcal{F}}{\partial \pi_y} + \frac{\partial \mathcal{F}}{\partial t} \\ &= \frac{1}{pq} \frac{\partial H}{\partial \pi_x} \frac{\partial \mathcal{F}}{\partial x} + \frac{1}{pq} \frac{\partial H}{\partial \pi_y} \frac{\partial \mathcal{F}}{\partial y} - \frac{\partial H}{\partial x} \frac{\partial \mathcal{F}}{\partial \pi_x} - \frac{\partial H}{\partial y} \frac{\partial \mathcal{F}}{\partial \pi_y} + \frac{\partial \mathcal{F}}{\partial t} \end{aligned} \tag{4.18}$$

where we have applied Hamilton's equations of motion. $\dot{\mathcal{F}}$ can be written in the form (4.7) if we define the deformed Poisson bracket $\{, \}_{q,p}$ of two functions \mathcal{F} and \mathcal{G} of the phase-space variables x, y, π_x and π_y as follows

$$\{\mathcal{F}, \mathcal{G}\}_{q,p} = \frac{1}{pq} \frac{\partial \mathcal{G}}{\partial \pi_x} \frac{\partial \mathcal{F}}{\partial x} - \frac{\partial \mathcal{G}}{\partial x} \frac{\partial \mathcal{F}}{\partial \pi_x} + \frac{1}{pq} \frac{\partial \mathcal{G}}{\partial \pi_y} \frac{\partial \mathcal{F}}{\partial y} - \frac{\partial \mathcal{G}}{\partial y} \frac{\partial \mathcal{F}}{\partial \pi_y}. \tag{4.19}$$

As a result of this definition, the Poisson brackets of the phase-space variables will not be antisymmetric for $pq \neq 1$. The canonical Poisson bracket relations are

$$\{x, \pi_x\}_{q,p} = \{y, \pi_y\}_{q,p} = \frac{1}{pq} \quad \{\pi_x, x\}_{q,p} = \{\pi_y, y\}_{q,p} = -1. \tag{4.20}$$

If we identify these Poisson brackets with elements of a matrix whose inverse corresponds to the components of a symplectic two-form $\omega_{q,p}$, the symplectic two-form then takes the form $\omega_{q,p} = (pq + 1)(\delta\pi_x \wedge \delta x + \delta\pi_y \wedge \delta y)$, where we have used the commutation relations (3.3). We see that the usual symplectic structure is recovered for $pq = 1$.

To compute the deformed Poisson brackets (4.19) of functions of the phase-space variables (which we can write in terms of a formal power series), we need to take partial derivatives of products of functions \mathcal{G} and \mathcal{H} on $T^*Q_{q,p}$. For this purpose it is necessary to know the commutativity properties of such functions with the phase-space variables (or more precisely, with variations of the phase-space variables). If we denote the phase-space variables by \bar{z}^i , then we can specify these commutativity properties by

$$\mathcal{G}\delta\bar{z}^i = \delta\bar{z}^i \mathcal{O}_{ji}^{(\mathcal{G})}. \tag{4.21}$$

$\mathcal{O}_{ij}^{(\mathcal{G})}$ can be computed using (4.14) and the commutation relations of sections 1 and 3. Although the usual Leibniz rule is assumed to hold for the exterior derivative δ , it does not necessarily apply for derivatives with respect to the phase space variables. Rather, from $\delta(\mathcal{G}\mathcal{H}) = \delta\mathcal{G}\mathcal{H} + \mathcal{G}\delta\mathcal{H}$ and equation (4.21), one gets [10]

$$\frac{\partial(\mathcal{G}\mathcal{H})}{\partial\bar{z}^i} = \frac{\partial\mathcal{G}}{\partial\bar{z}^i}\mathcal{H} + \mathcal{O}_{ij}^{(\mathcal{G})} \frac{\partial\mathcal{H}}{\partial\bar{z}^j}. \tag{4.22}$$

4.2. Example 2

The second Lagrangian we consider is first order in time derivatives and its bosonic and fermionic analogues have been studied long ago [8]. It is

$$L = \frac{x\dot{y} - qy\dot{x}}{1 + pq}. \tag{4.23}$$

(A similar Lagrangian was examined in [13] but there a different differential calculus was used.) Up to a total time derivative, L is equivalent to $x\dot{y}$. The Lagrangian (4.23) has the property that when we restrict to the case $q = p$, it is invariant under a subset of the $GL_{q,p}(2)$ transformations defined in equations (1.3) and (1.4). This subset is the one parameter deformation of the special linear group in two dimensions, standardly denoted by $SL_q(2)$, and it is obtained by setting

$$q = p \quad \text{and} \quad \det_q[T] = 1 \tag{4.24}$$

where

$$\det_q[T] \equiv AD - qBC.$$

Here one notes that $\det_q[T]$ so defined commutes with all matrix elements A, B, C and D when $q = p$, and therefore can be identified with the c-number ‘one’.

By taking the variational derivative of L we can get two different answers, depending upon whether we use case (a) or (b) commutation relations:

$$\begin{aligned} (1 + pq)\delta L &= \delta x\dot{y} + x\delta\dot{y} - q\delta y\dot{x} - qy\delta\dot{x} \\ &= \begin{cases} pq\delta x\dot{y} + q\delta\dot{y}x - \delta\dot{x}y - pq^2\delta y\dot{x} & \text{case (a)} \\ (1/pq)\delta x\dot{y} + q\delta\dot{y}x - \delta\dot{x}y - (1/p)\delta y\dot{x} & \text{case (b)}. \end{cases} \end{aligned} \tag{4.25}$$

In either case, however, we obtain the trivial equations of motion $\dot{x} = \dot{y} = 0$, and as a result the Lagrangian vanishes ‘on mass shell’.

The equations for the canonical momenta correspond to primary constraints (in the sense of Dirac), and they have the same form for both case (a) and (b),

$$\phi_x \equiv \pi_x + \frac{y}{1 + pq} \approx 0 \quad \text{and} \quad \phi_y \equiv \pi_y - \frac{qx}{1 + pq} \approx 0. \quad (4.26)$$

As a result, the commutativity properties of π_x and π_y (and their variations) are identical to the commutativity properties of $-y/(1 + pq)$ and $qx/(1 + pq)$ (and their variations), respectively.

It is easy to check that the Hamiltonian for this system is zero, or more precisely, it is a linear combination of the constraints (4.26). Now variations δH of H are different for cases (a) and (b). From equation (4.5),

$$\delta H = -\delta x \frac{\partial L}{\partial x} - \delta y \frac{\partial L}{\partial y} + \frac{s^2 q^2}{p} \delta \pi_x \dot{x} + \frac{s^2}{p} \delta \pi_y \dot{y} \quad (4.27)$$

where s can take two values: $s = p$ and $s = 1/q$. The former corresponds to case (a) and the latter corresponds to case (b). As a result of (4.27), Hamilton's equations of motion will be different for the two cases:

$$\begin{aligned} \dot{\pi}_x &= -\frac{\partial H}{\partial x} & \dot{x} &= \frac{p}{s^2 q^2} \frac{\partial H}{\partial \pi_x} \\ \dot{\pi}_y &= -\frac{\partial H}{\partial y} & \dot{y} &= \frac{p}{s^2} \frac{\partial H}{\partial \pi_y}. \end{aligned} \quad (4.28)$$

If \mathcal{F} is an arbitrary function of x , y , π_x , π_y and t , then its time derivative $\dot{\mathcal{F}}$ can be written in the form (4.7) if we define the deformed Poisson bracket $\{, \}_{q,p}$ of two functions \mathcal{F} and \mathcal{G} on the phase space as follows

$$\{\mathcal{F}, \mathcal{G}\}_{q,p} = \frac{p}{s^2 q^2} \frac{\partial \mathcal{G}}{\partial \pi_x} \frac{\partial \mathcal{F}}{\partial x} - \frac{\partial \mathcal{G}}{\partial x} \frac{\partial \mathcal{F}}{\partial \pi_x} + \frac{p}{s^2} \frac{\partial \mathcal{G}}{\partial \pi_y} \frac{\partial \mathcal{F}}{\partial y} - \frac{\partial \mathcal{G}}{\partial y} \frac{\partial \mathcal{F}}{\partial \pi_y}. \quad (4.29)$$

Then the Poisson brackets of the phase-space variables will be

$$\begin{aligned} \{x, \pi_x\}_{q,p} &= p/s^2 q^2 & \{y, \pi_y\}_{q,p} &= p/s^2 \\ \{\pi_x, x\}_{q,p} &= \{\pi_y, y\}_{q,p} & &= -1. \end{aligned} \quad (4.30)$$

Applying these Poisson brackets, we see that the constraints are not first class (neither for case (a) nor for case (b)), since

$$\{\phi_x, \phi_y\}_{q,p} = 1/s \quad \{\phi_y, \phi_x\}_{q,p} = -1/sq. \quad (4.31)$$

(Here we assume $pq \neq -1$.)

To eliminate the constraints ϕ_x and ϕ_y , we can apply the analogue of the Dirac procedure and replace deformed Poisson brackets $\{, \}_{q,p}$ by deformed Dirac brackets $\{, \}_{q,p}^*$. We define the latter by

$$\{\mathcal{F}, \mathcal{G}\}_{q,p}^* = \{\mathcal{F}, \mathcal{G}\}_{q,p} + s(q\{\mathcal{F}, \phi_x\}_{q,p}\{\phi_y, \mathcal{G}\}_{q,p} - \{\mathcal{F}, \phi_y\}_{q,p}\{\phi_x, \mathcal{G}\}_{q,p}) \quad (4.32)$$

where \mathcal{F} and \mathcal{G} are arbitrary functions on $T^*Q_{q,p}$. With this definition, it follows that Dirac brackets of the constraints ϕ_x and ϕ_y with any functions on the phase space vanish. We can now eliminate the constraints from the theory by working on a reduced phase space. We can take the reduced phase space to be the original quantum plane $Q_{q,p}$ parametrized by x

and y . If F and G are functions on the reduced phase space, their Dirac brackets simplify to

$$\begin{aligned} \{F, G\}_{q,p}^* &= s(q\{F, \pi_x\}_{q,p}\{\pi_y, G\}_{q,p} - \{F, \pi_y\}_{q,p}\{\pi_x, G\}_{q,p}) \\ &= \frac{p}{s} \left(-\frac{1}{q} \frac{\partial F}{\partial x} \frac{\partial G}{\partial y} + \frac{\partial F}{\partial y} \frac{\partial G}{\partial x} \right). \end{aligned} \quad (4.33)$$

We then see that the Dirac brackets between the coordinates x and y do not vanish, and further, are not antisymmetric. Instead,

$$\{x, y\}_{q,p}^* = -\frac{p}{sq} \quad \{y, x\}_{q,p}^* = \frac{p}{s}. \quad (4.34)$$

The Dirac bracket relations (4.34) are preserved under $SL_q(2)$ transformations (cf equations (1.3) and (4.24)). Such transformations are therefore *canonical*.

Instead of being antisymmetric, the brackets (4.34) have the property

$$\{x, y\}_{q,p}^* = -(1/q)\{y, x\}_{q,p}^*. \quad (4.35)$$

Equation (4.35) holds for both case (a) and case (b). From (4.34) the symplectic two-form $\omega_{q,p}$ for the theory is proportional to the two-form on $Q_{q,p}$:

$$\omega_{q,p} = \frac{s}{p}(pq + 1)\delta x \wedge \delta y.$$

5. Concluding remarks

In section 3 and the appendix we were able to construct two different differential calculi on $TQ_{q,p}$. The significance of having two different calculi is not evident, and there seems to be no reason for preferring one over the other. From them, it appears that starting from a single Lagrangian one may, in principle, derive two distinct dynamical systems. Furthermore, it may be possible to construct more differential calculi, and hence more dynamical systems, if we start with more general ansätze than the ones we used.

Although we developed the Lagrangian and Hamiltonian formalisms for particles moving on $Q_{q,p}$, we were unable to find non-trivial dynamical systems for arbitrary values of p and q which were consistent with the commutation relations (2.3). Furthermore, the Lagrangians we examined in section 4 were not central elements of the algebra generated by x , y , \dot{x} and \dot{y} . They thus cannot be identified with c-numbers. (However on mass shell, the Lagrangian (4.22) was a c-number, namely zero.) The interpretation of Feynman path integrals using such Lagrangians thus becomes problematic.

Nevertheless, a consistent canonical quantization of these systems may be possible, even though the Lagrangians are not c-numbers. The quantization procedure, however, is not uniquely determined. For example 2 we want to replace x and y , by some new non-commuting operators \mathbf{x} and \mathbf{y} , where the latter satisfy (1.1) in the limit $\hbar \rightarrow 0$. In the limit of $q = p = 1$, we should recover the usual quantization. For example 2 a possible quantization could mean replacing Dirac brackets (4.34) by the relation

$$\frac{1}{q}\mathbf{x}\mathbf{y} - \mathbf{y}\mathbf{x} = -\frac{i\hbar p}{sq}. \quad (5.1)$$

x and y are then deformed creation and annihilation operators. Thus for this particular example x and y are indeed the classical limit of deformed creation and annihilation operators. But this result is not general as it was obtained specifically from Lagrangian (4.23). (For algebras with more than two elements (excluding the central element), the introduction of central terms as in equation (5.1) may cause a break down of associativity. In order to preserve associativity, certain criteria have to be satisfied. For a discussion, see [14].)

Finally, we remark that although the Lagrangian (4.23) is invariant under $SL_q(2)$ transformations when $q = p$, and these transformations correspond to a canonical symmetry, we do not know how to apply Noether's theorem in this case to find the analogue of the symmetry generators (except for the case of $q = p = 1$, where the $SL(2)$ generators are 1 and i times binomials of x and y). The reason is that to apply Noether's theorem, we must be able to write the symmetry transformation (1.3) in infinitesimal form, and this is a non-trivial problem when $q = p \neq 1$. Yet $SL_q(2)$ (actually, $SU_q(2)$) generators have been constructed from binomials of deformed creation and annihilation operators [7].

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Appendix

Here we derive the commutation relations of x with $\delta\dot{y}$, y with $\delta\dot{x}$, \dot{x} with δy and \dot{y} with δx . For this, we define a 4×4 matrix $[F]$, whose elements are c-numbers, and make the ansatz

$$\begin{pmatrix} x\delta\dot{y} \\ y\delta\dot{x} \\ \dot{x}\delta y \\ \dot{y}\delta x \end{pmatrix} = [F] \begin{pmatrix} \delta\dot{y}x \\ \delta\dot{x}y \\ \delta y\dot{x} \\ \delta x\dot{y} \end{pmatrix}. \quad (\text{A.1})$$

The matrix elements can be determined (i) by demanding that no secondary conditions on combinations of z^i and δz^i result from commuting z^i and δz^i through the commutation relations (1.1), (2.1) and (2.2); (ii) from the identities (3.2); and (iii) by demanding that the relations (A.1) are preserved under the $GL_{q,p}(2)$ transformations (1.3).

(i) By multiplying $\delta\dot{x}$ and $\delta\dot{y}$ on the right of $xy - qyx = 0$, we find

$$F_{11} = q \quad \text{and} \quad F_{21} = 0 \quad (\text{A.2})$$

along with the consistency conditions

$$\begin{aligned} F_{12}(p - F_{22}) &= 0 \\ (1 - pq)(pF_{13} + 1 - pq) &= F_{12}F_{23} \\ p(1 - pq)(F_{14} + 1 - pq) &= F_{12}F_{24} \\ F_{22} + p(F_{23} - pq) + F_{24}pq &= 0. \end{aligned} \quad (\text{A.3})$$

By multiplying δx and δy on the right of the commutation relations (2.1), we find that

$$F_{31} = 0 \quad \text{and} \quad F_{41} = 0 \quad (\text{A.4})$$

along with the consistency conditions

$$\begin{aligned}
 F_{42} + pF_{32} &= 0 \\
 F_{42}F_{24} &= F_{42}F_{23} = 0 \\
 F_{42}(F_{22} - p^2q) &= 0 \\
 1 - F_{43} - pF_{33} &= 0 \\
 p - F_{44} - pF_{34} &= 0 \\
 F_{32} + pF_{33} + pq(F_{34} - 1) &= 0 \\
 F_{42} + pF_{43} + pqF_{44} - p &= 0.
 \end{aligned} \tag{A.5}$$

(ii) Upon substituting the ansatz (A.1) into the identities (3.2), we get the following additional conditions

$$\begin{aligned}
 F_{12} + F_{32} &= qp - 1 \\
 F_{13} + F_{33} &= q \\
 F_{14} + F_{34} &= qp - 1 \\
 F_{22} + F_{42} &= p \\
 F_{23} + F_{43} &= 0 \\
 F_{24} + F_{44} &= p.
 \end{aligned} \tag{A.6}$$

For arbitrary q and p , there are two solutions to equations (A.3), (A.5) and (A.6). One of them yields the following matrix $[F]$

$$[F] = \begin{pmatrix} q & 0 & (1/p)(pq - 1) & pq - 1 \\ 0 & p^2q & 0 & 0 \\ 0 & pq - 1 & 1/p & 0 \\ 0 & p(1 - pq) & 0 & p \end{pmatrix}. \tag{A.7}$$

However, it is not hard to show that the commutation relations resulting from (A.7) do not fulfil (iii), that is they are not preserved under $GL_{q,p}(2)$ transformations (1.3), and we shall thus not consider this solution further.

On the other hand, $GL_{q,p}(2)$ transformations do preserve the other solution

$$[F] = \begin{pmatrix} q & qp - 1 & q(1 - s/p) & pq - 2 + s/p \\ 0 & p & qs - 1 & p - s \\ 0 & 0 & qs/p & 1 - s/p \\ 0 & 0 & 1 - qs & s \end{pmatrix}. \tag{A.8}$$

To determine the quantity s , we can (i) multiply δx on the right of $\dot{x}\dot{y} - q\dot{y}\dot{x} = 0$. This yields

$$(1 - qs)(s/p - 1) = 0. \tag{A.9}$$

s is thus allowed to take two different values, $s = p$ and $s = 1/q$, and they correspond to the two cases (a) and (b), respectively. The commutation relations given in equations (3.12) and (3.13) result.

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